# Introduction to Programming: Lecture 11

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# The binary tree datatype

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data Btree a = Nil | Node (Btree a) a (Btree a)
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     (Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil))
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Node (Node Nil 4 Nil) 6
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```

```
        6
       / \
      4   3
         / \
        2   5
```

# Levels

- List nodes level by level and from left to right within each level.

```
    4
   / \
  2   5
 / \
1   3
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[4,2,5,1,3]
```

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       / \
      2   5
     / \
    1   3
```

```
[4,2,5,1,3]
```

```
mylevels: Btree a -> [[a]]
mylevels Nil = []
mylevels (Node tl x tr) =
    [x]:(join (mylevels tl) (mylevels tr))

level t = concat (mylevels t)
```

# Levels

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# Search trees

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  - Values in the left subtree are smaller than the current node
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- Two search trees for values `[1,2,3,4,5,7]`

```
     4                  3
    / \                / \
   2   5              2   5
  / \   \            /   / \
 1   3   7          1   4   7
```

## Search trees

- In a search tree
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```
        4                      3
       / \                    / \
      2   5                  2   5
     / \   \                /   / \
    1   3   7              1   4   7
```

- Search Trees in Haskell

```
data Ord a => Stree a = Nil | Node (Stree a) a (Stree a
 deriving (Eq, Show)
```

# Search trees

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      4                  3
     / \                / \
    2   5              2   5
   / \   \            /   / \
  1   3   7          1   4   7
```

- Search Trees in Haskell

```haskell
data Ord a => Stree a = Nil | Node (Stree a) a (Stree a
 deriving (Eq, Show)
```

  - Need `Ord a` to compare values
  - No gurantee of being a search tree!

# Search trees ...

- Is it a search tree?

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```
isstree:: Ord a => (Stree a) -> Bool
isstree Nil = True
isstree (Node tl y tr)
     = (isstree tl) && (isstree tr) &&
           (maxt tl < y) && (y < (mint tr))
```

# Search trees ...

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mint (Node Nil v Nil) = v
mint (Node tl v tr) = min (mint tl) (min v (mint tr))
```

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```
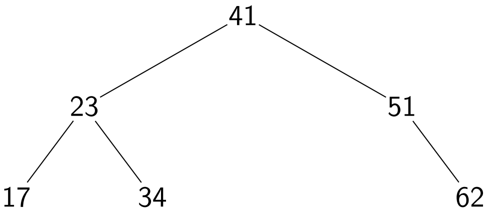

- In how many ways is the above program incorrect?

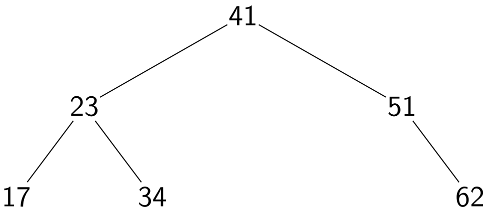
# Search trees . . .

- Searching for a value



# Search trees . . .

- Searching for a value
  Searching for 34



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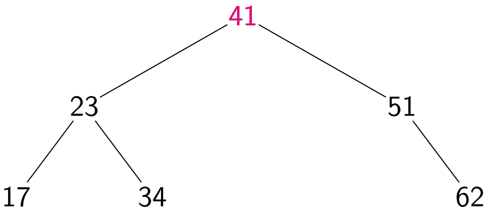

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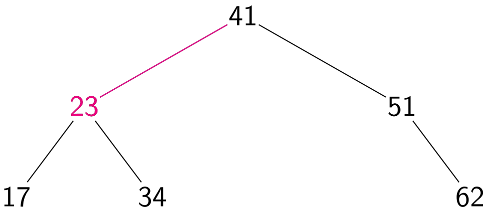

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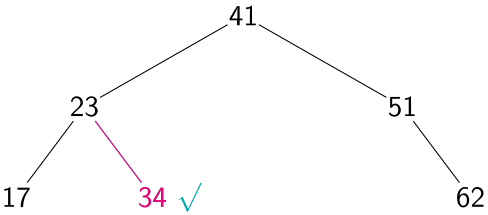
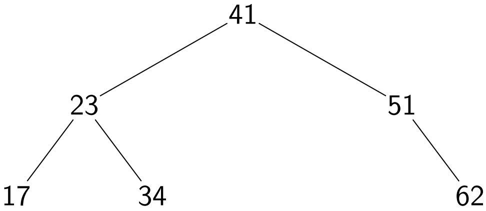


# Search trees . . .

- Searching for a value
  Searching for 49



# Search trees ...

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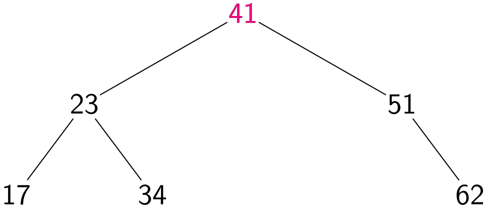

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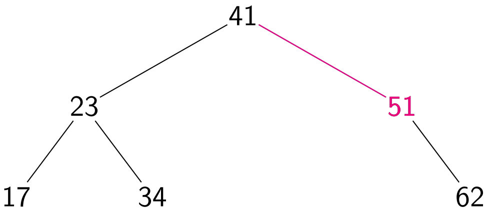

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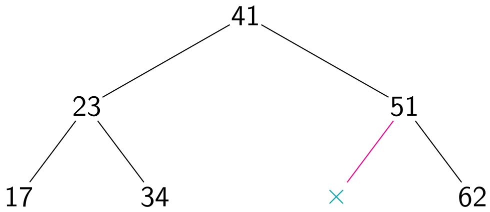

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  - If the value is found, report Yes

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- Searching for a value $v$
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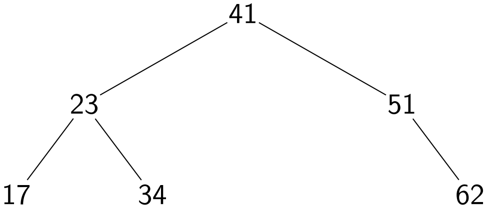
searchtree :: Ord a => (Stree a) -> a -> Bool
searchtree Nil v = False
searchtree (Node tl y tr) v
  | v == y    = True
  | v < y     = searchtree tl v
  | otherwise = searchtree tr v
```

# Search trees, inserting a value

- Insert a value



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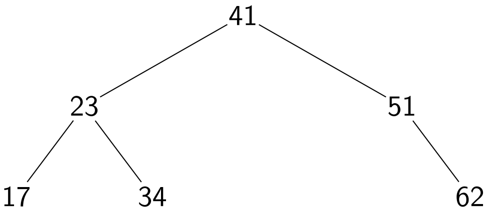

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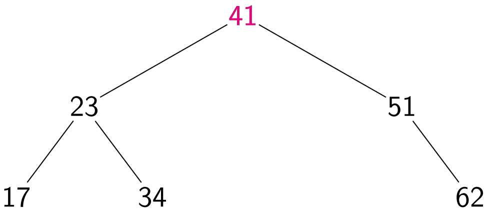

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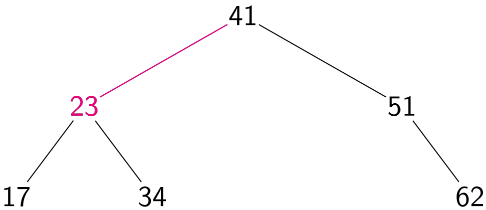

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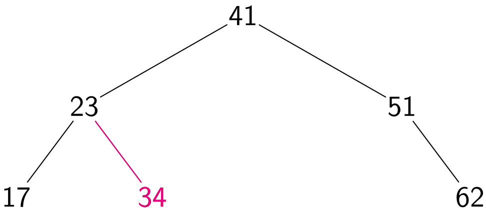

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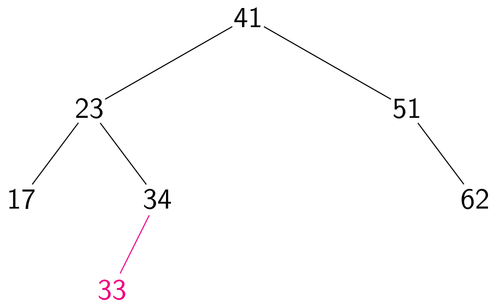

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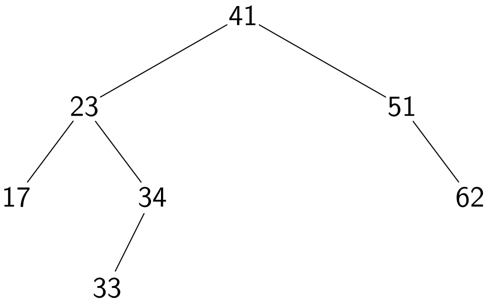

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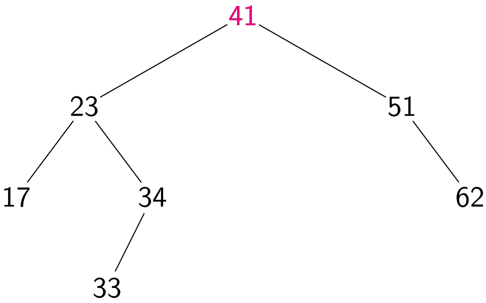

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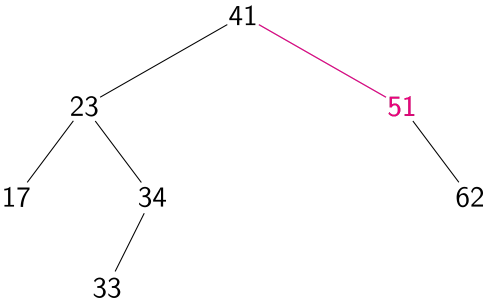

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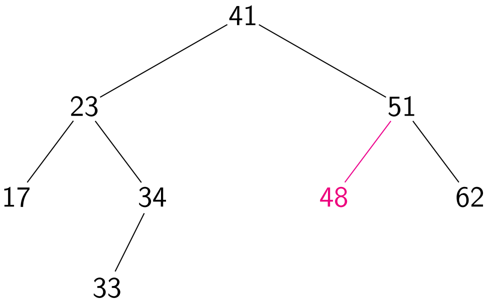
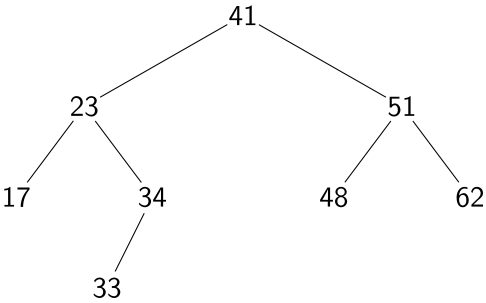


# Search trees, inserting a value

- Insert a value
  Insert 17



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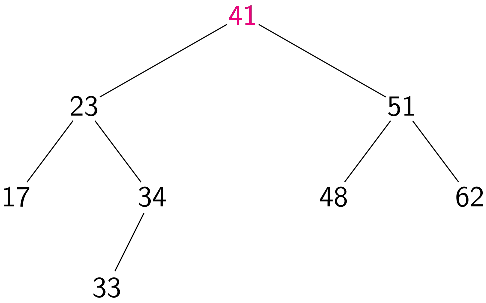

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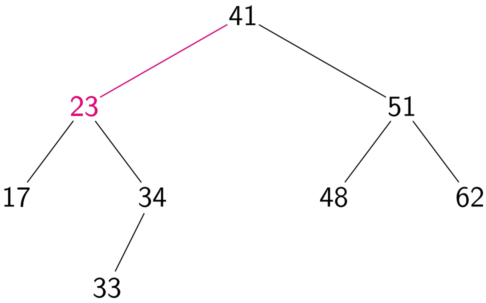

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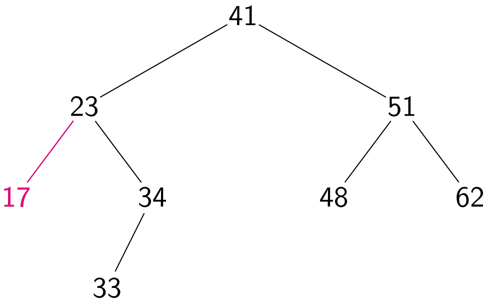

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    - If right child exists, insert $v$ in right subtree
    - Otherwise, add a right child with value $v$
- Worst case: Number of steps is equal to the longest path from the root to a leaf

# Inserting into a search tree

- To insert a value, search for it to identify where it should go

```
inserttree :: Ord a => Stree a  -> a -> Stree a
inserttree Nil v = Node  Nil v Nil
inserttree (Node  tl y tr) v
  | v == y    = Node  tl y tr
  | v < y     = Node (inserttree tl v) y  tr
  | otherwise = Node  tl y (inserttree tr v)
```

- `inserttree` returns the tree with the value inserted.

# Search trees, deleting a value

- Deleting $v$ from a tree

## Search trees, deleting a value

- Deleting $v$ from a tree
- If $v$ does not match current node, inductively delete from left or right subtree

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- What if $v$ does match?

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        y == v
       /   \
      x     z
     / \   / \
    t1 t2 t3 t4
```

- What value should replace `y`?

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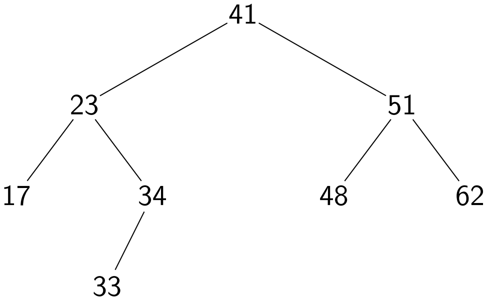
  t1 t2 t3 t4
```

- What value should replace `y`?
- Cannot blindly shift up `x` or `z`

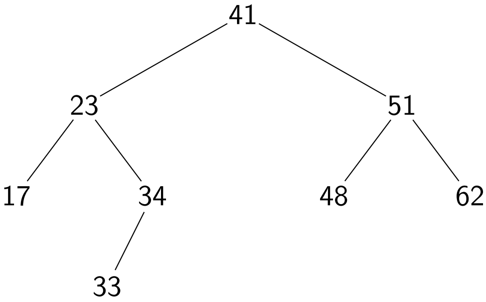
# Search trees, deleting a value . . .

- Delete a value



# Search trees, deleting a value . . .

- Delete a value
  Delete 41



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- Delete a value
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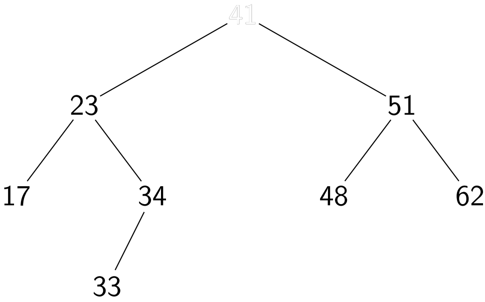
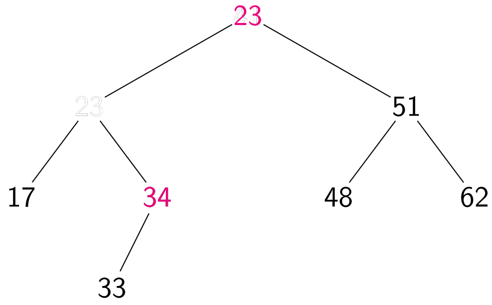


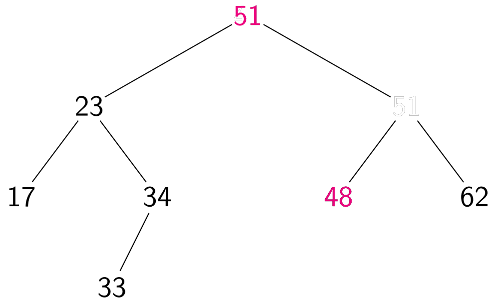
# Search trees, deleting a value . . .

- Delete a value
  Delete 41
  Cannot shift up 23



# Search trees, deleting a value . . .

- Delete a value
  Delete 41
  Cannot shift up 51



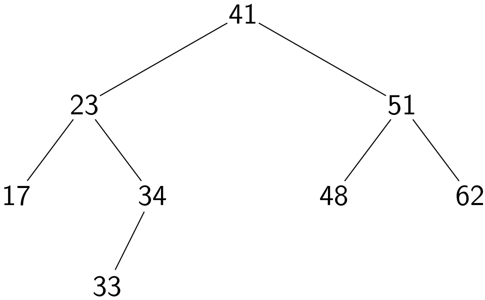
# Search trees, deleting a value . . .

```
        y == v
       /   \
      x     z
     / \   / \
    t1 t2 t3 t4
```

- Cannot blindly shift up `x` or `z`
- Need to move up a value that is bigger than left and smaller than right
    - Move up maximum value in left subtree . . .
    - . . . or minimum value in right subtree

# Search trees, deleting a value . . .

- Delete a value



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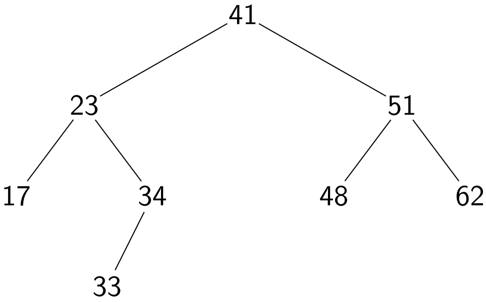

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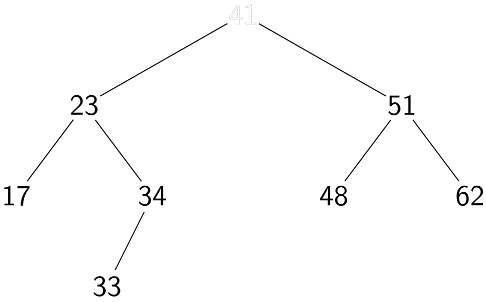

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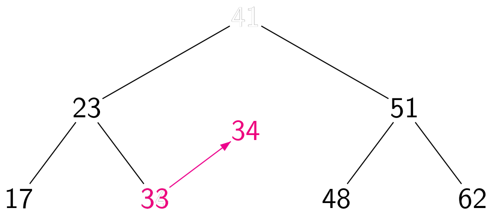
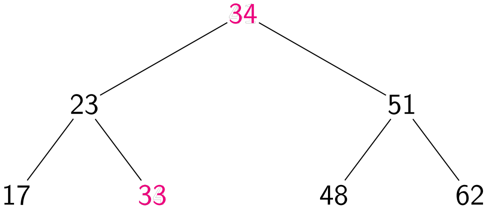
  Remove maximum value in left subtree, 34



# Search trees, deleting a value . . .

- Delete a value
  Delete 41
  Remove maximum value in left subtree, 34
  . . . and use it to replace 41



# Search trees . . .

- Deleting the maximum value in a search tree

## Search trees . . .

- Deleting the maximum value in a search tree
- Keep going right till you run out of values
    - Rightmost value has no right subtree
    - Replace rightmost value by its left subtree

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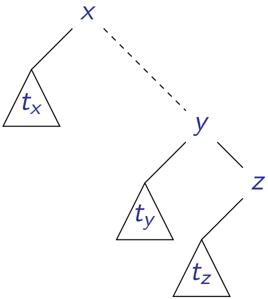

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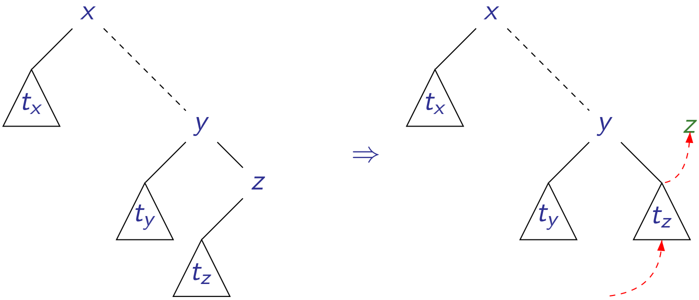

# Deleting maximum value in a search tree ...

- deletemax

```
deletemax :: Ord a => Stree a  -> (a ,Stree a)

-- We are at rightmost value
deletemax (Node  t1 y Nil) = (y,t1)

-- We are not yet at rightmost value
deletemax (Node t1 y t2) = (z, Node t1 y tz)
  where (z,tz) = deletemax t2
```

# Deleting maximum value in a search tree ...

- `deletemax`

```
deletemax :: Ord a => Stree a  -> (a ,Stree a)

-- We are at rightmost value
deletemax (Node  t1 y Nil) = (y,t1)

-- We are not yet at rightmost value
deletemax (Node t1 y t2) = (z, Node t1 y tz)
  where (z,tz) = deletemax t2
```

- Note that `deletemax` returns the maximum value and the modified tree

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- Worst case: Number of steps is equal to the longest path from the root to a leaf

# Deleting from a search tree

- deletetree — deletes a given value from the given tree and returns the resulting tree

```
deletetree :: Ord a => Stree a  -> a -> Stree a
deletetree Nil v = Nil
deletetree (Node tl y  tr) v
  | v < y   = Node (deletetree tl v)  y tr
  | v > y   = Node  tl  y (deletetree tr v)

-- In all cases below, we must have v == y

deletetree (Node Nil y tr) v   = tr
deletetree (Node  tl y tr) v = Node  tx x tr
  where (x,tx) = deletemax tl
```

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- Inserting values in ascending or descending order results in highly skewed tree

```
          6
         /
        5
       /
      4
     /
    3
   /
  2
 /
1
```

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- However, it is not easy to maintain size-balance.

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- Height is still logarithmic in size [Adelson-Velskii, Landis]
- Somewhat easier to maintain.

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```
    4
   / \
  2   5
 / \   \
1   3   6
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- A height balanced tree that is not weight balanced.

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- Grows like the Fibonacci numbers, exponentially.
- $S(h) \geq 2^{h/1.44}$ or equivalently

$$h(T) \leq 1.44 log(s(T))$$

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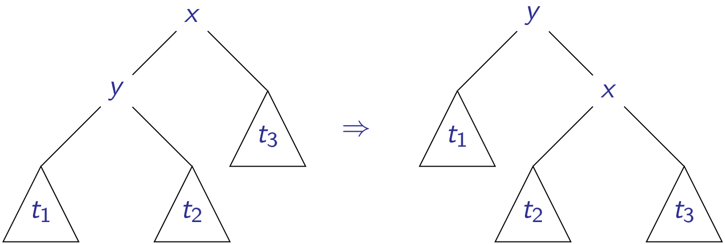

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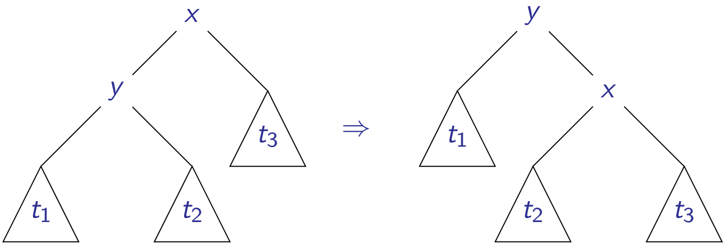


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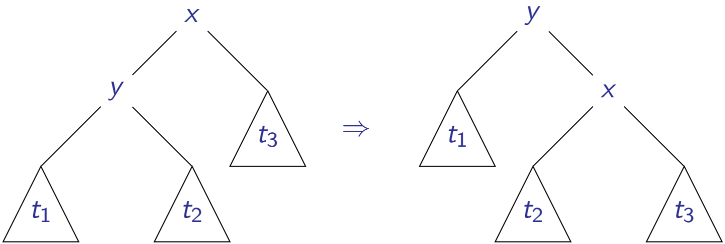


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  rotateright (Node (Node t1 y t2) x t3) =
                                  Node t1 y (Node t2 x t3)
  ```
  ```

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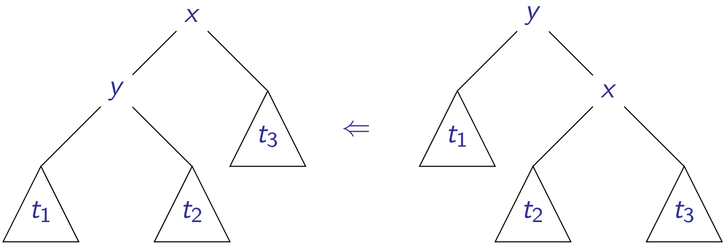


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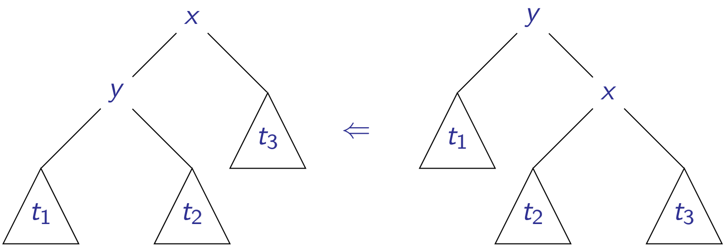


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  rotateleft (Node t1 y (Node t2 x t3)) =
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  ```
  ```

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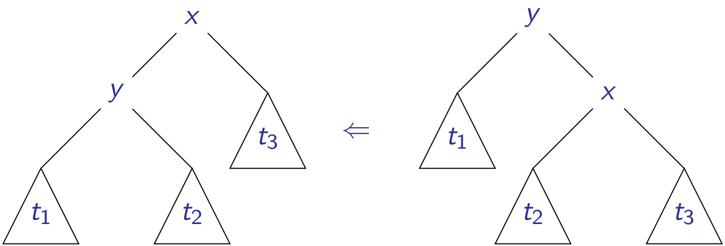


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