Introduction to Programming: Lecture 10

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Evaluating postfix expressions

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- Another example:

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- Another example:

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- Another example:

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- Another example:

  `2 3 + 7 2 + -`

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Evaluating postfix expressions

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  `3 5 8 * +`

  `43`

- Another example:

  `2 3 + 7 2 + -`

  `-4`

- Keep a stack of numbers.
  - If you see a number, push it on to the stack.
  - If you seen an operator, remove the top two elements from the stack, evaluate and push the result on the stack.

Programming the calculator in Haskell

- The structure of the program:

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- The structure of the program:
  - A module to manage stacks.
  - A module that handles expressions and their evalutation.

The Stack module

- Methods `empty`, `push`, `pop` and `isempty`.

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```
data Stack a = Empty | Stack a (Stack a)
empty :: Stack a
empty = Empty

push :: a -> Stack a -> Stack a
push x st = Stack x st

pop :: Stack a -> (a, Stack a)
pop (Stack x st) = (x, st)

isempty :: Stack a -> Bool
isempty Empty = True
isempty _     = False
```

The Stack module

- Methods `empty`, `push`, `pop` and `isempty`.
- As general a type as possible for `Stack`.

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isempty :: Stack a -> Bool
isempty Empty = True
isempty _     = False
```

- It looks very much like a `list`!

The Stack module via lists

```haskell
data Stack a = Stack [a]

empty :: Stack a
empty = Stack []

push :: a -> Stack a -> Stack a
push x (Stack ls) = Stack (x:ls)

pop :: Stack a -> (a, Stack a)
pop (Stack (x:ls)) = (x, Stack ls)

isempty :: Stack a -> Bool
isempty (Stack []) = True
isempty _          = False
```

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```
module Stack(Stack(),empty,push,pop,isempty) where
data Stack a =  ...

empty :: Stack a
...

push :: a -> Stack a -> Stack a
...

pop :: Stack a -> (a, Stack a)
...

isempty :: Stack a -> Bool
...
```

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data Token = Val Int | Op Char
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- How do we represent it in Haskell?
- We use the word `Token` to denote an element of the expression

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type Expr = [Token]
```

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- An one step evaluation function:

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evalStep  st (Val i) = push i st
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    | c == '+' = push (v2 + v1) st2
    | c == '-' = push (v2 - v1) st2
    | c == '*' = push (v2 * v1) st2
            where
            (v1,st1) = pop st
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             where
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- How to iterate this and evaluate the entire expression?

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evaluate exp = evalExp empty exp
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- Now, we can use any implementation of the `Stack` and it works identically.

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```
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isemptyq :: (Queue a) -> Bool
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- Required operations

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emptyq :: Queue a
addq :: a -> (Queue a) -> (Queue a)
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```

- Again, represent a queue using a list

```
data Queue a = Qu [a]

emptyq = Qu []
addq x (Qu xs) = (Qu (xs ++ [x]))
removeq (Qu (x:xs)) = (x,Qu xs)
isempty (Qu l) = (l == [])
```

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removeq (Qu xs) = ((last x),Qu (init xs))
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Now, removing an element takes $O(n)$ time.

Adding and removing n elements could take $O(n^2)$ time

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- Split queue and store the rear in reversed order in a reversed

  Represent $[q_1, q_2, \ldots, q_n]$
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- What happens if the first list is empty?
- If the first list is empty, reverse the second list on to the first list and then remove the first element.

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- `removeq` takes from first list, reversing elements from second list into first list if necessary

```
removeq (Nuqu (x:xs) ys)  = (x,Nuqu xs ys)
removeq (Nuqu [] ys)  = removeq (Nuqu (reverse ys) [])
```

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```

- If we add n elements, we get a queue `Nuqu [] [qn,...,q1]`
  - Next `removeq` takes $O(n)$ time to reverse the second list
  - After one `removeq`, we have `Nuqu [q2,...,qn] []`
  - Next $n-1$ `removeq` operations take time $O(1)$!

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- Each element can be touched only four times.
- In any sequence of N instructions at most N elements are involved.
- Any sequence of N instructions can take only $O(N)$ steps!

The `Set` datastructure

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data Eq a => Set a = Set [a]
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search x (Set y)= elem x y
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insert x (Set s)
  | elem x  (Set s)  = Set s
  | otherwise = Set (x:s)
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search x (Set y)= elem x y
insert x (Set s)
  | elem x  (Set s)  = Set s
  | otherwise = Set (x:s)
delete x (Set s)  = Set (filter (/= x) s)
```

Complexity of the operations

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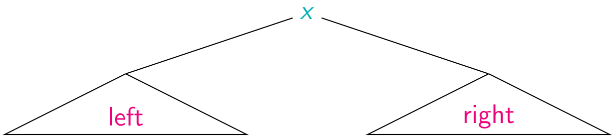
- `search` takes linear time.
- `insert` takes linear time.
- `delete` takes linear time.
- A sequence of N operations can take $O(N^2)$ time.
- We can do better if the elements of the type `a` can be ordered.

A datatype for binary trees

- Trees are recursive datatypes
- A tree is either
  - Empty
  - Or is a node containing a value and two trees



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data Btree a = Nil | Node (Btree a) a (Btree a)
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```

- `Nil` and `Node` are the constructors.
- `Nil` represents the empty tree.
- A nonempty tree (identified by the constructor `Node`) has three parts
  - A left (sub-)tree
  - A value
  - A right (sub-)tree

Examples of trees

```
Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil)
```

Examples of trees

```
Node (Node Nil 2 Nil) 3 (Node Nil 5 Nil)

        3
       / \
      2   5
```

Examples of trees

```
Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil)

        3
       / \
      2   5

Node (Node Nil 4 Nil) 6
     (Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil))
```

Examples of trees

```
Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil)

        3
       / \
      2   5

Node (Node Nil 4 Nil) 6
     (Node (Node Nil 2 Nil)  3 (Node Nil 5 Nil))

        6
       / \
      4   3
         / \
        2   5
```

Binary Trees ...

- What about

```
      4
     / \
    2   5
   / \
  1   3
```

Binary Trees ...

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```
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`Node (Node (Node Nil 1 Nil) 2 (Node Nil 3 Nil))`
`4 (Node Nil 5 Nil)`

Functions on Binary trees

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```
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size Nil = 0
size (Node tl x tr) = 1 + (size tl) + (size tr)
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- `height` – Longest path from the root to a leaf

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size (Node tl x tr) = 1 + (size tl) + (size tr)
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- height – Longest path from the root to a leaf

```
height :: Btree a -> Int
height Nil = 0
height (Node t1 x tr) =
      1 + (max  (height tl) (height tr))
```

Levels

- List nodes level by level and from left to right within each level.

```
    4
   / \
  2   5
 / \
1   3
```

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```
    4
   / \
  2   5
 / \
1   3
```

```
[4,2,5,1,3]
```